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Some matrices associated with the split decomposition for a Q-polynomial distance-regular graph

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Abstract

We consider a Q-polynomial distance-regular graph Γ with vertex set X and diameter $D \geq 3$. For $\mu, \nu \in \{\downarrow, \uparrow\}$ we define a direct sum decomposition of the standard module $V = \mathbb{C}X$, called the (μ, ν)-split decomposition. For this decomposition we compute the complex conjugate and transpose of the associated primitive idempotents. Now fix $b, \beta \in \mathbb{C}$ such that $b \neq 1$ and assume Γ has classical parameters (D, b, α, β) with $\alpha = b - 1$. Under this assumption Ito and Terwilliger displayed an action of the q-tetrahedron algebra $\boxtimes_q$ on the standard module of Γ. To describe this action they defined eight matrices in $\mathrm{Mat}_X(\mathbb{C})$, called

$$A, \quad A^*, \quad B, \quad B^*, \quad K, \quad K^*, \quad \Phi, \quad \Psi.$$

For each matrix in the above list we compute the transpose and complex conjugate. Using this information we compute the transpose and complex conjugate for each generator of $\boxtimes_q$ on V.

1. Introduction

We consider a Q-polynomial distance-regular graph Γ with vertex set X and diameter $D \geq 3$ (see Section 2 for formal definitions). Let V denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by X and whose entries are in $\mathbb{C}$. We call V the *standard module* of Γ. For $\mu, \nu \in \{\downarrow, \uparrow\}$ we define a direct sum decomposition of V called the (μ, ν)*–split decomposition*. In this decomposition the components are indexed by ordered pairs of integers (i, j) $(0 \leq i, j \leq D)$. For $0 \leq i, j \leq D$ we let $E^{\mu\nu}_{i,j}$ denote the projection onto component

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(i, j) of the decomposition. We show that $E^{\mu\nu}_{i,j}$ is real. We show that $(E^{\downarrow\downarrow}_{i,j})^t = E^{\uparrow\uparrow}_{D-i,D-j}$ and $(E^{\downarrow\uparrow}_{i,j})^t = E^{\uparrow\downarrow}_{D-i,D-j}$, where t denotes transpose. Now fix $b, \beta \in \mathbb{C}$ such that $b \neq 1$ and assume Γ has classical parameters (D, b, α, β) with $\alpha = b - 1$. Fix $q \in \mathbb{C}$ such that $b = q^2$. Under this assumption Ito and Terwilliger displayed an action of the q-tetrahedron algebra $\boxtimes_q$ on the standard module of Γ [11]. To describe this action they defined eight matrices in $\mathrm{Mat}_X(\mathbb{C})$, called

$$A, \quad A^*, \quad B, \quad B^*, \quad K, \quad K^*, \quad \Phi, \quad \Psi.$$

For each matrix in the above list we compute the transpose and complex conjugate. Concerning the transpose, we show that each of A, A^*, Φ, Ψ is symmetric and that $B^t = B^*$ and $K^t = K^{*-1}$. Concerning the complex conjugate, recall from the above assumption that $q \in \mathbb{C}$ satisfies $b = q^2$. Define $q' = -q$ and note that $(q')^2 = b$. There are two cases to consider. For $b > 1$ we have $\overline{q} = q$ and $\overline{q'} = q'$, where $\overline{}$ denotes complex conjugation. For $b < -1$ we have $q \in i\mathbb{R}$ so $\overline{q} = q'$. For each matrix S in the above list let S' denote the corresponding matrix associated with q'. We show that for each matrix S from the above list, S is real if $b > 1$ and $\overline{S} = S'$ if $b < -1$. Using the above information we compute the transpose and complex conjugate for each generator of $\boxtimes_q$ on V.

2. The Terwilliger algebra of a distance-regular graph; preliminaries

In this section we review some definitions and basic concepts concerning the Terwilliger algebra of a distance-regular graph. For more background information we refer the reader to [1,3,9,15].

Let X denote a nonempty finite set. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{C}$. Let $V = \mathbb{C}X$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by X and whose entries are in $\mathbb{C}$. We observe that $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication. We call V the *standard module*. We endow V with the Hermitian form $\langle , \rangle$ that satisfies $\langle u, v\rangle = u^t\overline{v}$ for $u, v \in V$. Observe that $\langle , \rangle$ is positive definite. We call this form the *standard Hermitian form* on V. Observe that for $B \in \mathrm{Mat}_X(\mathbb{C})$,

$$\langle Bu, v\rangle = \langle u, \overline{B}^t v\rangle \quad u, v \in V. \tag{1}$$

For all $y \in X$, let $\hat{y}$ denote the element of V with a 1 in the y coordinate and 0 in all other coordinates. Observe that $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for V.

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X and edge set R. Let ∂ denote the path-length distance function for Γ, and set $D := \max\{\partial(x, y) \mid x, y \in X\}$. We call D the *diameter* of Γ. We say that Γ is *distance-regular* whenever for all integers h, i, j $(0 \le h, i, j \le D)$ and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\{z \in X \mid \partial(x, z) = i, \partial(z, y) = j\}|$$

is independent of x and y. The p^h_{ij} are called the *intersection numbers* of Γ. We use the abbreviation $c_i = p^i_{1,i-1} (1 \le i \le D)$, $b_i = p^i_{1,i+1} (0 \le i \le D - 1)$, $a_i = p^i_{1i} (0 \le i \le D)$.

For the rest of this paper we assume that Γ is distance-regular with diameter $D \ge 3$.

We recall the Bose–Mesner algebra of Γ. For $0 \le i \le D$ let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with xy entry

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \partial(x, y) = i \\ 0, & \text{if } \partial(x, y) \neq i \end{cases} \quad (x, y \in X).$$

We call A_i the ith *distance matrix* of Γ. A_1 is called the *adjacency matrix* of Γ. We observe that (i) $A_0 = I$; (ii) $\sum_{i=0}^{D} A_i = J$; (iii) $\overline{A_i} = A_i$ $(0 \le i \le D)$; (iv) $A_i^t = A_i$ $(0 \le i \le D)$; (v) $A_i A_j = \sum_{h=0}^{D} p_{ij}^h A_h$ $(0 \le i, j \le D)$, where I (resp. J) denotes the identity matrix (resp. all 1's matrix) in $\mathrm{Mat}_X(\mathbb{C})$. Using these facts we find that $A_0, A_1, \ldots, A_D$ form a basis for a commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$. We call M the *Bose–Mesner algebra* of Γ. It turns out that A_1 generates M [1, p. 190]. By [3, p. 45], M has a second basis $E_0, E_1, \ldots, E_D$ such that (i) $E_0 = |X|^{-1} J$; (ii) $\sum_{i=0}^{D} E_i = I$; (iii) $\overline{E_i} = E_i$ $(0 \le i \le D)$; (iv) $E_i^t = E_i$ $(0 \le i \le D)$; (v) $E_i E_j = \delta_{ij} E_i$ $(0 \le i, j \le D)$. We call $E_0, E_1, \ldots, E_D$ the *primitive idempotents* of Γ.

We recall the eigenvalues of Γ. Since $E_0, E_1, \ldots, E_D$ form a basis for M there exist complex scalars $\theta_0, \theta_1, \ldots, \theta_D$ such that $A_1 = \sum_{i=0}^{D} \theta_i E_i$. Observe that $A_1 E_i = E_i A_1 = \theta_i E_i$ for $0 \le i \le D$. We call θ_i the *eigenvalue* of Γ associated with E_i $(0 \le i \le D)$. Since A_1 is real symmetric, the eigenvalues $\theta_0, \theta_1, \ldots, \theta_D$ are in $\mathbb{R}$. Observe that $\theta_0, \theta_1, \ldots, \theta_D$ are mutually distinct since A_1 generates M. Observe that

$$V = E_0 V + E_1 V + \cdots + E_D V \qquad \text{(orthogonal direct sum).} \tag{2}$$

For $0 \le i \le D$ the space $E_i V$ is the eigenspace of A_1 associated with θ_i.

We recall the Krein parameters. Let $\circ$ denote the entrywise product in $\mathrm{Mat}_X(\mathbb{C})$. Observe that $A_i \circ A_j = \delta_{ij} A_i$ for $0 \le i, j \le D$, so M is closed under $\circ$. Thus there exist complex scalars $q_{ij}^h (0 \le h, i, j \le D)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q_{ij}^h E_h \quad (0 \le i, j \le D).$$

By [2, p. 170], q_{ij}^h is real and nonnegative for $0 \le h, i, j \le D$. The q_{ij}^h are called the *Krein parameters*. The graph Γ is said to be *Q-polynomial* (with respect to the given ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents) whenever for $0 \le h, i, j \le D$, $q_{ij}^h = 0$ (resp. $q_{ij}^h \neq 0$) whenever one of h, i, j is greater than (resp. equal to) the sum of the other two [3, p. 59]. See [1,4,5,13,14] for more information on the Q-polynomial property. From now on we assume that Γ is Q-polynomial with respect to $E_0, E_1, \ldots, E_D$.

We recall the dual Bose–Mesner algebra of Γ. Fix a vertex $x \in X$. We view x as a "base vertex". For $0 \le i \le D$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with yy entry

$$(E_i^*)_{yy} = \begin{cases} 1, & \text{if } \partial(x, y) = i \\ 0, & \text{if } \partial(x, y) \neq i \end{cases} \quad (y \in X). \tag{3}$$

We call E_i^* the ith *dual idempotent* of Γ with respect to x [15, p. 378]. We observe that (i) $\sum_{i=0}^{D} E_i^* = I$; (ii) $\overline{E_i^*} = E_i^* (0 \le i \le D)$; (iii) $E_i^{*t} = E_i^* (0 \le i \le D)$; (iv) $E_i^* E_j^* = \delta_{ij} E_i^* (0 \le i, j \le D)$. By these facts $E_0^*, E_1^*, \ldots, E_D^*$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$. We call M^* the *dual Bose–Mesner algebra* of Γ with respect to x [15, p. 378]. For $0 \le i \le D$ let $A_i^* = A_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with yy entry $(A_i^*)_{yy} = |X|(E_i)_{xy}$ for $y \in X$. Then $A_0^*, A_1^*, \ldots, A_D^*$ form a basis for M^* [15, p. 379]. Moreover (i) $A_0^* = I$; (ii) $\overline{A_i^*} = A_i^*$ $(0 \le i \le D)$; (iii) $A_i^{*t} = A_i^*$ $(0 \le i \le D)$;

(iv) $A_i^* A_j^* = \sum_{h=0}^{D} q_{ij}^h A_h^*$ $(0 \le i, j \le D)$ [15, p. 379]. We call $A_0^*, A_1^*, \ldots, A_D^*$ the *dual distance matrices* of Γ with respect to x. A_1^* is called the *dual adjacency matrix* of Γ with respect to x. The matrix A_1^* generates M^* [15, Lemma 3.11].

We recall the dual eigenvalues of Γ. Since $E_0^*, E_1^*, \ldots, E_D^*$ form a basis for M^* and since A_1^* is real, there exist real scalars $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ such that $A_1^* = \sum_{i=0}^{D} \theta_i^* E_i^*$. Observe that $A_1^* E_i^* = E_i^* A_1^* = \theta_i^* E_i^*$ for $0 \le i \le D$. We call θ_i^* the *dual eigenvalue* of Γ associated with $E_i^* (0 \le i \le D)$. Observe that $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ are mutually distinct since A_1^* generates M^*.

We recall the subconstituents of Γ. From (3) we find

$$E_i^* V = \mathrm{span}\{\hat{y} | y \in X, \quad \partial(x, y) = i\} \qquad (0 \le i \le D). \tag{4}$$

By (4) and since $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for V we find

$$V = E_0^* V + E_1^* V + \cdots + E_D^* V \qquad \text{(orthogonal direct sum)}. \tag{5}$$

For $0 \le i \le D$ the space $E_i^* V$ is the eigenspace of A_1^* associated with θ_i^*. We call $E_i^* V$ the ith *subconstituent* of Γ with respect to x.

We recall the subconstituent algebra of Γ. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M and M^*. We call T the *subconstituent algebra* (or *Terwilliger algebra*) of Γ with respect to x [15, Definition 3.3]. We observe that T is generated by A_1, A_1^*. We observe that T has finite dimension. Moreover T is semi-simple since it is closed under the conjugate transpose map [7, p. 157]. See [6,8,15–17] for more information on the subconstituent algebra.

Until further notice we adopt the following notation.

Notation 2.1. We assume that $\Gamma = (X, R)$ is a distance-regular graph with diameter $D \ge 3$. We assume that Γ is Q-polynomial with respect to the ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents. We fix $x \in X$ and write $A_1^* = A_1^*(x)$, $E_i^* = E_i^*(x)$ $(0 \le i \le D)$, $T = T(x)$. We use the abbreviation $V = \mathbb{C}X$.

We recall some useful results on T-modules. With reference to Notation 2.1, by a *T-module* we mean a subspace $W \subseteq V$ such that $BW \subseteq W$ for all $B \in T$. Let W denote a T-module. Then W is said to be *irreducible* whenever W is nonzero and W contains no T-modules other than 0 and W. Let W denote a T-module and let W' denote a T-module contained in W. Then the orthogonal complement of W' in W is a T-module [8, p. 802]. It follows that each T-module is an orthogonal direct sum of irreducible T-modules. In particular V is an orthogonal direct sum of irreducible T-modules. Let W denote an irreducible T-module. By the *endpoint* of W we mean $\min\{i | 0 \le i \le D,\ E_i^* W \ne 0\}$. By the *diameter* of W we mean $|\{i | 0 \le i \le D,\ E_i^* W \ne 0\}| - 1$. By the *dual endpoint* of W we mean $\min\{i | 0 \le i \le D,\ E_i W \ne 0\}$. By the *dual diameter* of W we mean $|\{i | 0 \le i \le D, E_i W \ne 0\}| - 1$. The diameter of W is equal to the dual diameter of W [13, Corollary 3.3].

Lemma 2.2 (*[15, Lemmas 3.4, 3.9, 3.12]*). *With reference to Notation 2.1, let W denote an irreducible T-module with endpoint ρ, dual endpoint τ, and diameter d. Then ρ, τ, d are nonnegative integers such that $\rho + d \le D$ and $\tau + d \le D$. Moreover the following* (i)–(iv) *hold.*

(i) $E_i^* W \ne 0$ *if and only if* $\rho \le i \le \rho + d$ $(0 \le i \le D)$.
(ii) $W = \sum_{h=0}^{d} E_{\rho+h}^* W$ (*orthogonal direct sum*).
(iii) $E_i W \ne 0$ *if and only if* $\tau \le i \le \tau + d$ $(0 \le i \le D)$.
(iv) $W = \sum_{h=0}^{d} E_{\tau+h} W$ (*orthogonal direct sum*).

Lemma 2.3 (*[5, Lemmas 5.1, 7.1]*). *With reference to Notation 2.1, let W denote an irreducible T-module with endpoint ρ, dual endpoint τ, and diameter d. Then the following* (i), (ii) *hold.*

(i) $2\rho + d \geq D$.
(ii) $2\tau + d \geq D$.

We finish this section with a comment.

Lemma 2.4 (*[12, Lemma 3.3]*). *With reference to Notation 2.1, let W denote an irreducible T-module with endpoint ρ, dual endpoint τ, and diameter d. Then for $\mu, \nu \in \{\downarrow, \uparrow\}$ we have*

$$W = \sum_{h=0}^{d} W_h^{\mu\nu} \quad (\textit{direct sum}),$$

where for $0 \leq h \leq d$,

$$\begin{aligned}
W_h^{\downarrow\downarrow} &= (E^*_\rho W + \cdots + E^*_{\rho+h} W) \cap (E_\tau W + \cdots + E_{\tau+d-h} W),\\
W_h^{\uparrow\downarrow} &= (E^*_{\rho+d-h} W + \cdots + E^*_{\rho+d} W) \cap (E_\tau W + \cdots + E_{\tau+d-h} W),\\
W_h^{\downarrow\uparrow} &= (E^*_\rho W + \cdots + E^*_{\rho+h} W) \cap (E_{\tau+h} W + \cdots + E_{\tau+d} W),\\
W_h^{\uparrow\uparrow} &= (E^*_{\rho+d-h} W + \cdots + E^*_{\rho+d} W) \cap (E_{\tau+h} W + \cdots + E_{\tau+d} W).
\end{aligned}$$

3. The displacement and split decompositions of the standard module

In this section we recall the displacement decompositions and the four split decompositions for the standard module and discuss their basic properties and connections. We start with a definition.

Definition 3.1 (*[18, Definition 4.1]*). With reference to Notation 2.1, let W denote an irreducible T-module. By the *displacement of W of the first kind* (resp. *second kind*) we mean the integer $\rho + \tau + d - D$ (resp. $\rho - \tau$), where ρ, τ, d denote respectively the endpoint, dual endpoint, and diameter of W.

Lemma 3.2. *With reference to Notation 2.1, let W denote an irreducible T-module. Then the following* (i), (ii) *hold.*

(i) *Let η denote the displacement of W of the first kind. Then* $0 \leq \eta \leq D$.
(ii) *Let ζ denote the displacement of W of the second kind. Then* $-D \leq \zeta \leq D$.

Proof. (i) This is just [18, Lemma 4.2].
(ii) Recall that $0 \leq \rho, \tau \leq D$ by Lemma 2.2. The result follows. □

Definition 3.3. With reference to Notation 2.1, for $0 \leq \eta \leq D$ let V_η denote the subspace of V spanned by the irreducible T-modules for which η is the displacement of the first kind. Observe that V_η is a T-module. By [18, Lemma 4.4] we have $V = \sum_{\eta=0}^{D} V_\eta$ (orthogonal direct sum). For $0 \leq \eta \leq D$ we define a matrix $\phi_\eta \in \mathrm{Mat}_X(\mathbb{C})$ so that

$$\begin{aligned}
&(\phi_\eta - I) V_\eta = 0,\\
&\phi_\eta V_\xi = 0 \quad \text{if } \eta \neq \xi \ (0 \leq \xi \leq D).
\end{aligned}$$

In other words ϕ_η is the projection from V onto V_η. We note that $V_\eta = \phi_\eta V$.

The following result is immediate from Definition 3.3.

Lemma 3.4. *With reference to Notation 2.1,*

$$V = \sum_{\eta=0}^{D} \phi_\eta V \quad (\textit{orthogonal direct sum}). \tag{6}$$

Moreover for $0 \leq \eta \leq D$ the subspace $\phi_\eta V$ of V is spanned by the irreducible T-modules for which η is the displacement of the first kind.

Definition 3.5. We call the sum (6) the *displacement decomposition of V of the first kind* with respect to x.

Lemma 3.6. *With reference to Notation 2.1 and Definition 3.3, the following* (i), (ii) *hold.*

(i) $\sum_{\eta=0}^{D} \phi_\eta = I$.
(ii) $\phi_\eta \phi_\xi = \delta_{\eta\xi} \phi_\eta$ $(0 \leq \eta, \xi \leq D)$.

Proof. Immediate from Definition 3.3. $\square$

We have been discussing the displacement of the first kind. We now do something similar for the displacement of the second kind.

Definition 3.7. With reference to Notation 2.1, for $-D \leq \zeta \leq D$ let V_ζ denote the subspace of V spanned by the irreducible T-modules for which ζ is the displacement of the second kind. Observe that V_ζ is a T-module. By [18, Lemma 4.4] $V = \sum_{\zeta=-D}^{D} V_\zeta$ (orthogonal direct sum). For $-D \leq \zeta \leq D$ we define a matrix $\psi_\zeta \in \mathrm{Mat}_X(\mathbb{C})$ so that

$$(\psi_\zeta - I)V_\zeta = 0,$$
$$\psi_\zeta V_\xi = 0 \quad \text{if } \zeta \neq \xi \ (-D \leq \xi \leq D).$$

In other words ψ_ζ is the projection from V onto V_ζ. We note that $V_\zeta = \psi_\zeta V$.

The following result is immediate from Definition 3.7.

Lemma 3.8. *With reference to Notation 2.1,*

$$V = \sum_{\zeta=-D}^{D} \psi_\zeta V \quad (\textit{orthogonal direct sum}). \tag{7}$$

Moreover for $-D \leq \zeta \leq D$ the subspace $\psi_\zeta V$ of V is spanned by the irreducible T-modules for which ζ is the displacement of the second kind.

Definition 3.9. We call the sum (7) the *displacement decomposition of V of the second kind* with respect to x.

Lemma 3.10. *With reference to Notation 2.1 and Definition 3.7, the following* (i), (ii) *hold.*

(i) $\sum_{\zeta=-D}^{D} \psi_\zeta = I$.
(ii) $\psi_\zeta \psi_\xi = \delta_{\zeta\xi} \psi_\zeta$ $(-D \leq \zeta, \xi \leq D)$.

Proof. Immediate from Definition 3.7. $\square$

We now recall the split decompositions of V.

Definition 3.11 (*[11, Definition 10.1]*). With reference to Notation 2.1, for $-1 \leq i, j \leq D$ we define

$$V_{i,j}^{\downarrow\downarrow} = (E_0^*V + \cdots + E_i^*V) \cap (E_0V + \cdots + E_jV),$$
$$V_{i,j}^{\uparrow\downarrow} = (E_D^*V + \cdots + E_{D-i}^*V) \cap (E_0V + \cdots + E_jV),$$
$$V_{i,j}^{\downarrow\uparrow} = (E_0^*V + \cdots + E_i^*V) \cap (E_DV + \cdots + E_{D-j}V),$$
$$V_{i,j}^{\uparrow\uparrow} = (E_D^*V + \cdots + E_{D-i}^*V) \cap (E_DV + \cdots + E_{D-j}V).$$

In each of the above four equations we interpret the right-hand side as being 0 if $i = -1$ or $j = -1$.

Lemma 3.12. *With reference to Notation 2.1, the following hold for $0 \leq i \leq D$.*

$$V_{i,D}^{\downarrow\downarrow} = E_0^*V + \cdots + E_i^*V, \qquad V_{D,i}^{\downarrow\downarrow} = E_0V + \cdots + E_iV, \tag{8}$$
$$V_{i,D}^{\uparrow\downarrow} = E_D^*V + \cdots + E_{D-i}^*V, \qquad V_{D,i}^{\uparrow\downarrow} = E_0V + \cdots + E_iV, \tag{9}$$
$$V_{i,D}^{\downarrow\uparrow} = E_0^*V + \cdots + E_i^*V, \qquad V_{D,i}^{\downarrow\uparrow} = E_DV + \cdots + E_{D-i}V, \tag{10}$$
$$V_{i,D}^{\uparrow\uparrow} = E_D^*V + \cdots + E_{D-i}^*V, \qquad V_{D,i}^{\uparrow\uparrow} = E_DV + \cdots + E_{D-i}V. \tag{11}$$

Proof. Immediate from (2) and (5), and Definition 3.11. $\square$

Definition 3.13 (*[11, Definition 10.2]*). With reference to Notation 2.1 and Definition 3.11, for $\mu, \nu \in \{\downarrow, \uparrow\}$ and $0 \leq i, j \leq D$ we have $V_{i-1,j}^{\mu\nu} \subseteq V_{i,j}^{\mu\nu}$ and $V_{i,j-1}^{\mu\nu} \subseteq V_{i,j}^{\mu\nu}$. Therefore

$$V_{i-1,j}^{\mu\nu} + V_{i,j-1}^{\mu\nu} \subseteq V_{i,j}^{\mu\nu}.$$

Referring to the above inclusion, we define $\tilde{V}_{i,j}^{\mu\nu}$ to be the orthogonal complement of the left-hand side in the right-hand side; that is

$$\tilde{V}_{i,j}^{\mu\nu} = (V_{i-1,j}^{\mu\nu} + V_{i,j-1}^{\mu\nu})^{\perp} \cap V_{i,j}^{\mu\nu}.$$

The following result is a mild generalization of [18, Theorem 5.7].

Proposition 3.14. *With reference to Notation 2.1 and Definition 3.13, for $\mu, \nu \in \{\downarrow, \uparrow\}$ and $0 \leq i, j \leq D$ we have*

$$V_{i,j}^{\mu\nu} = \sum_{r=0}^{i} \sum_{s=0}^{j} \tilde{V}_{r,s}^{\mu\nu} \quad (\textit{direct sum}). \tag{12}$$

Proof. For $\mu = \downarrow$, $\nu = \downarrow$ this is just [18, Theorem 5.7]. For general values of μ, ν the argument is essentially the same. $\square$

Corollary 3.15 (*[11, Lemma 10.3]*). *With reference to Notation 2.1 and Definition 3.13, for $\mu, \nu \in \{\downarrow, \uparrow\}$ we have*

$$V = \sum_{i=0}^{D}\sum_{j=0}^{D} \tilde{V}_{i,j}^{\mu\nu} \quad (\textit{direct sum}). \tag{13}$$

Definition 3.16. We call the sum (13) the (μ, ν)-*split decomposition* of V with respect to x.

Remark 3.17. The decomposition (13) is not orthogonal in general.

Corollary 3.18. *With reference to Notation 2.1 and Definition 3.13, the following* (i), (ii) *hold.*

(i) *For $0 \le i \le D$ the dimension of $E_i^* V$ is equal to each of*

$$\sum_{j=0}^{D} \dim \tilde{V}_{i,j}^{\downarrow\downarrow}, \qquad \sum_{j=0}^{D} \dim \tilde{V}_{D-i,j}^{\uparrow\downarrow},$$
$$\sum_{j=0}^{D} \dim \tilde{V}_{i,j}^{\downarrow\uparrow}, \qquad \sum_{j=0}^{D} \dim \tilde{V}_{D-i,j}^{\uparrow\uparrow}.$$

(ii) *For $0 \le j \le D$ the dimension of $E_j V$ is equal to each of*

$$\sum_{i=0}^{D} \dim \tilde{V}_{i,j}^{\downarrow\downarrow}, \qquad \sum_{i=0}^{D} \dim \tilde{V}_{i,D-j}^{\downarrow\uparrow},$$
$$\sum_{i=0}^{D} \dim \tilde{V}_{i,j}^{\uparrow\downarrow}, \qquad \sum_{i=0}^{D} \dim \tilde{V}_{i,D-j}^{\uparrow\uparrow}.$$

Proof. To get $\dim E_i^* V = \sum_{j=0}^{D} \dim \tilde{V}_{i,j}^{\downarrow\downarrow}$, set $j = D$ in (12), evaluate the result by using the equation on the left in (8), and use induction on i. The other equations are similarly obtained. □

Lemma 3.19 (*[12, Lemma 4.6]*). *With reference to Notation 2.1, let W denote an irreducible T-module with endpoint ρ, dual endpoint τ, and diameter d. Then the following* (i)–(iv) *hold for $0 \le h \le d$ and $0 \le i, j \le D$.*

(i) $W_h^{\downarrow\downarrow} \subseteq \tilde{V}_{i,j}^{\downarrow\downarrow}$ *if and only if $i = \rho + h$ and $j = \tau + d - h$.*
(ii) $W_h^{\uparrow\downarrow} \subseteq \tilde{V}_{i,j}^{\uparrow\downarrow}$ *if and only if $i = D - \rho - d + h$ and $j = \tau + d - h$.*
(iii) $W_h^{\downarrow\uparrow} \subseteq \tilde{V}_{i,j}^{\downarrow\uparrow}$ *if and only if $i = \rho + h$ and $j = D - \tau - h$.*
(iv) $W_h^{\uparrow\uparrow} \subseteq \tilde{V}_{i,j}^{\uparrow\uparrow}$ *if and only if $i = D - \rho - d + h$ and $j = D - \tau - h$.*

In the following two theorems we describe the relationships between the displacement decompositions and the split decompositions.

Theorem 3.20. *With reference to Notation 2.1, the following* (i), (ii) *hold for $0 \le \eta \le D$.*

(i) $\phi_\eta V = \sum \tilde{V}_{i,j}^{\downarrow\downarrow}$*, where the sum is over all ordered pairs i, j such that $0 \le i, j \le D$ and $i + j = D + \eta$.*

(ii) $\phi_\eta V = \sum \tilde{V}_{i,j}^{\uparrow\uparrow}$, *where the sum is over all ordered pairs* i, j *such that* $0 \le i, j \le D$ *and* $i + j = D - \eta$.

Proof. Part (i) is just [18, Theorem 6.2(i)]. Part (ii) is similarly proved. □

Theorem 3.21. *With reference to Notation* 2.1, *the following* (i), (ii) *hold for* $-D \le \zeta \le D$.

(i) $\psi_\zeta V = \sum \tilde{V}_{i,j}^{\downarrow\uparrow}$, *where the sum is over all ordered pairs* i, j *such that* $0 \le i, j \le D$ *and* $i + j = D + \zeta$.
(ii) $\psi_\zeta V = \sum \tilde{V}_{i,j}^{\uparrow\downarrow}$, *where the sum is over all ordered pairs* i, j *such that* $0 \le i, j \le D$ *and* $i + j = D - \zeta$.

Proof. Similar to the proof of Theorem 3.20. □

Corollary 3.22. *With reference to Notation* 2.1, *the following* (i), (ii) *hold for* $0 \le i, j \le D$.

(i) $\tilde{V}_{i,j}^{\downarrow\downarrow} = 0$ *if* $i + j < D$.
(ii) $\tilde{V}_{i,j}^{\uparrow\uparrow} = 0$ *if* $i + j > D$.

Proof. (i) This is just [18, Theorem 6.2(ii)].
(ii) Immediate from Lemma 3.4 and Theorem 3.20(ii). □

In [12] we showed that with respect to the standard Hermitian form the $(\downarrow, \downarrow)$-split decomposition (resp. $(\downarrow, \uparrow)$-split decomposition) and the $(\uparrow, \uparrow)$-split decomposition (resp. $(\uparrow, \downarrow)$ -split decomposition) are dual in the following sense.

Theorem 3.23 (*[12, Theorem 4.8]*). *With reference to Notation* 2.1 *and Definition* 3.13, *the following* (i), (ii) *hold for* $0 \le i, j, r, s \le D$.

(i) $\tilde{V}_{i,j}^{\downarrow\downarrow}$ *and* $\tilde{V}_{r,s}^{\uparrow\uparrow}$ *are orthogonal unless* $i + r = D$ *and* $j + s = D$.
(ii) $\tilde{V}_{i,j}^{\downarrow\uparrow}$ *and* $\tilde{V}_{r,s}^{\uparrow\downarrow}$ *are orthogonal unless* $i + r = D$ *and* $j + s = D$.

4. The matrices $E_{i,j}^{\downarrow\downarrow}$, $E_{i,j}^{\uparrow\downarrow}$, $E_{i,j}^{\downarrow\uparrow}$, $E_{i,j}^{\uparrow\uparrow}$

In this section we use the split decompositions to define the matrices $E_{i,j}^{\mu\nu}$ for $0 \le i, j \le D$ and $\mu, \nu \in \{\downarrow, \uparrow\}$. We then discuss some basic properties of these matrices. We start with a definition.

Definition 4.1. With reference to Notation 2.1, for $0 \le i, j \le D$ and for $\mu, \nu \in \{\downarrow, \uparrow\}$ we define $E_{i,j}^{\mu\nu} \in \mathrm{Mat}_X(\mathbb{C})$ so that

$$(E_{i,j}^{\mu\nu} - I)\tilde{V}_{i,j}^{\mu\nu} = 0,$$
$$E_{i,j}^{\mu\nu}\tilde{V}_{r,s}^{\mu\nu} = 0 \quad \text{if } (i, j) \ne (r, s)\ (0 \le r, s \le D).$$

In other words $E_{i,j}^{\mu\nu}$ is the projection from V onto $\tilde{V}_{i,j}^{\mu\nu}$. We note that

$$E_{i,j}^{\mu\nu} V = \tilde{V}_{i,j}^{\mu\nu}. \tag{14}$$

Lemma 4.2. *With reference to Notation* 2.1 *and Definition* 4.1, *the following* (i), (ii) *hold for* $\mu, \nu \in \{\downarrow, \uparrow\}$ *and* $0 \le i, j, r, s \le D$.

(i) $\sum_{i=0}^{D}\sum_{j=0}^{D} E_{i,j}^{\mu\nu} = I$.
(ii) $E_{i,j}^{\mu\nu} E_{r,s}^{\mu\nu} = \delta_{ir}\delta_{js} E_{i,j}^{\mu\nu}$.

Proof. Immediate from Corollary 3.15 and Definition 4.1. $\square$

Corollary 4.3. *With reference to Notation* 2.1 *and Definition* 4.1, *the following* (i), (ii) *hold for* $0 \le \eta \le D$.

(i) $\phi_\eta = \sum E_{i,j}^{\downarrow\downarrow}$, *where the sum is over all ordered pairs* i, j *such that* $0 \le i, j \le D$ *and* $i + j = D + \eta$.
(ii) $\phi_\eta = \sum E_{i,j}^{\uparrow\uparrow}$, *where the sum is over all ordered pairs* i, j *such that* $0 \le i, j \le D$ *and* $i + j = D - \eta$.

Proof. Combine Theorem 3.20 and Definition 4.1. $\square$

Corollary 4.4. *With reference to Notation* 2.1 *and Definition* 4.1, *the following* (i), (ii) *hold for* $-D \le \zeta \le D$.

(i) $\psi_\zeta = \sum E_{i,j}^{\downarrow\uparrow}$, *where the sum is over all ordered pairs* i, j *such that* $0 \le i, j \le D$ *and* $i + j = D + \zeta$.
(ii) $\psi_\zeta = \sum E_{i,j}^{\uparrow\downarrow}$, *where the sum is over all ordered pairs* i, j *such that* $0 \le i, j \le D$ *and* $i + j = D - \zeta$.

Proof. Combine Theorem 3.21 and Definition 4.1. $\square$

Corollary 4.5. *With reference to Notation* 2.1 *and Definition* 4.1, *the following* (i), (ii) *hold for* $0 \le i, j \le D$.

(i) $E_{i,j}^{\downarrow\downarrow} = 0$ *if* $i + j < D$.
(ii) $E_{i,j}^{\uparrow\uparrow} = 0$ *if* $i + j > D$.

Proof. Immediate from Corollary 3.22 and Definition 4.1. $\square$

Lemma 4.6. *With reference to Notation* 2.1 *and Definition* 3.13, *pick any* $\mu, \nu \in \{\downarrow, \uparrow\}$ *and* $0 \le i, j \le D$. *Then the following* (i), (ii) *hold for any* $v \in V$.

(i) $v \in V_{i,j}^{\mu\nu}$ *if and only if* $\overline{v} \in V_{i,j}^{\mu\nu}$.
(ii) $v \in \tilde{V}_{i,j}^{\mu\nu}$ *if and only if* $\overline{v} \in \tilde{V}_{i,j}^{\mu\nu}$.

Proof. (i) The result follows by Definition 3.11 and since E_i, E_i^* are real for $0 \le i \le D$.
(ii) Routine using Definition 3.13 and (i) above. $\square$

Theorem 4.7. *With reference to Notation* 2.1 *and Definition* 4.1, *the following* (i)–(iii) *hold for* $\mu, \nu \in \{\downarrow, \uparrow\}$ *and* $0 \le i, j \le D$.

(i) $\overline{E_{i,j}^{\mu\nu}} = E_{i,j}^{\mu\nu}$.
(ii) $(E_{i,j}^{\downarrow\downarrow})^t = E_{D-i,D-j}^{\uparrow\uparrow}$.
(iii) $(E_{i,j}^{\downarrow\uparrow})^t = E_{D-i,D-j}^{\uparrow\downarrow}$.

Proof. (i) By Definition 4.1 it suffices to show that $(\overline{E_{i,j}^{\mu\nu}} - I)\tilde{V}_{i,j}^{\mu\nu} = 0$ and $\overline{E_{i,j}^{\mu\nu}}\tilde{V}_{r,s}^{\mu\nu} = 0$ if $(i,j) \neq (r,s)$. To see $(\overline{E_{i,j}^{\mu\nu}} - I)\tilde{V}_{i,j}^{\mu\nu} = 0$, pick v in $\tilde{V}_{i,j}^{\mu\nu}$ and note that $\overline{v} \in \tilde{V}_{i,j}^{\mu\nu}$ by Lemma 4.6(ii). Now

$$
\begin{aligned}
(\overline{E_{i,j}^{\mu\nu}} - I)v &= \overline{(E_{i,j}^{\mu\nu} - I)\overline{v}} \\
&= 0
\end{aligned}
$$

so $(\overline{E_{i,j}^{\mu\nu}} - I)\tilde{V}_{i,j}^{\mu\nu} = 0$. Next we fix r, s $(0 \le r, s \le D)$ such that $(r,s) \neq (i,j)$ and show that $\overline{E_{i,j}^{\mu\nu}}\tilde{V}_{r,s}^{\mu\nu} = 0$. Pick v in $\tilde{V}_{r,s}^{\mu\nu}$ and note that $\overline{v} \in \tilde{V}_{r,s}^{\mu\nu}$ by Lemma 4.6(ii). Observe that

$$
\begin{aligned}
\overline{E_{i,j}^{\mu\nu}}v &= \overline{E_{i,j}^{\mu\nu}\overline{v}} \\
&= 0
\end{aligned}
$$

so $\overline{E_{i,j}^{\mu\nu}}\tilde{V}_{r,s}^{\mu\nu} = 0$ and the result follows.

(ii) We show that $(E_{D-i,D-j}^{\uparrow\uparrow})^t = E_{i,j}^{\downarrow\downarrow}$. By Definition 4.1 it suffices to show that $((E_{D-i,D-j}^{\uparrow\uparrow})^t - I)\tilde{V}_{i,j}^{\downarrow\downarrow} = 0$ and $(E_{D-i,D-j}^{\uparrow\uparrow})^t\tilde{V}_{r,s}^{\downarrow\downarrow} = 0$ if $(i,j) \neq (r,s)$. First we fix r, s $(0 \le r, s \le D)$ such that $(r,s) \neq (i,j)$ and show that $(E_{D-i,D-j}^{\uparrow\uparrow})^t\tilde{V}_{r,s}^{\downarrow\downarrow} = 0$. To do this it suffices to show that $\langle (E_{D-i,D-j}^{\uparrow\uparrow})^t\tilde{V}_{r,s}^{\downarrow\downarrow}, V\rangle = 0$. Observe that

$$
\begin{aligned}
\langle (E_{D-i,D-j}^{\uparrow\uparrow})^t\tilde{V}_{r,s}^{\downarrow\downarrow}, V\rangle &= \langle \tilde{V}_{r,s}^{\downarrow\downarrow}, E_{D-i,D-j}^{\uparrow\uparrow}V\rangle && \text{(by (1) and (i))} \\
&= \langle \tilde{V}_{r,s}^{\downarrow\downarrow}, \tilde{V}_{D-i,D-j}^{\uparrow\uparrow}\rangle && \text{(by (14))} \\
&= 0 && \text{(by Theorem 3.23(i))}
\end{aligned}
$$

so $(E_{D-i,D-j}^{\uparrow\uparrow})^t\tilde{V}_{r,s}^{\downarrow\downarrow} = 0$. To obtain $((E_{D-i,D-j}^{\uparrow\uparrow})^t - I)\tilde{V}_{i,j}^{\downarrow\downarrow} = 0$ combine Lemma 4.2(i) and our previous comments.

(iii) Similar to the proof of (ii). □

Corollary 4.8. *With reference to Notation* 2.1 *and Definition* 4.1*, the following* (i), (ii) *hold for* $0 \le \eta \le D$.

(i) $\overline{\phi_\eta} = \phi_\eta$.
(ii) $\phi_\eta^t = \phi_\eta$.

Proof. (i) Immediate from Corollary 4.3 and Theorem 4.7(i).
(ii) In the equation of Corollary 4.3(i) take the transpose and evaluate the result using Corollary 4.3(ii) and Theorem 4.7(ii). □

Corollary 4.9. *With reference to Notation* 2.1 *and Definition* 4.1*, the following* (i), (ii) *hold for* $-D \le \zeta \le D$.

(i) $\overline{\psi_\zeta} = \psi_\zeta$.
(ii) $\psi_\zeta^t = \psi_\zeta$.

Proof. Similar to the proof of Corollary 4.8. □

5. A restriction on the intersection numbers

For the rest of this paper we impose the following restriction on the intersection numbers of Γ.

Assumption 5.1. We fix $b, \beta \in \mathbb{C}$ such that $b \neq 1$, and assume Γ has classical parameters (D, b, α, β) with $\alpha = b - 1$. This means that the intersection numbers of Γ satisfy

$$c_i = b^{i-1}\frac{b^i - 1}{b - 1},$$
$$b_i = (\beta + 1 - b^i)\frac{b^{\mathrm{D}} - b^i}{b - 1}$$

for $0 \leq i \leq D$ [3, p. 193]. We remark that b is an integer and $b \neq 0$, $b \neq -1$ [3, Proposition 6.2.1]. For notational convenience we fix $q \in \mathbb{C}$ such that $b = q^2$. We note that q is nonzero and not a root of unity.

Remark 5.2. Referring to Assumption 5.1, the restriction $\alpha = b - 1$ implies that Γ is formally self-dual [3, Corollary 8.4.4]. Consequently there exists an ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents of Γ, with respect to which the Krein parameter q^h_{ij} is equal to the intersection number p^h_{ij} for $0 \leq h, i, j \leq D$. In particular Γ is Q-polynomial with respect to $E_0, E_1, \ldots, E_D$. We fix this ordering of the primitive idempotents for the rest of the paper.

With reference to Assumption 5.1, Ito and Terwilliger displayed an action of the q-tetrahedron algebra $\boxtimes_q$ on the standard module of Γ [11]. To describe this action they defined eight matrices in $\mathrm{Mat}_X(\mathbb{C})$, called

$$A, \quad A^*, \quad B, \quad B^*, \quad K, \quad K^*, \quad \Phi, \quad \Psi. \tag{15}$$

For each matrix in (15) we compute the transpose and complex conjugate. Using this information we compute the transpose and complex conjugate for each generator of $\boxtimes_q$.

6. The matrices A, A^*, B, B^*, K, K^*, Φ, Ψ

In this section we define the matrices (15) and discuss their properties. We begin with a lemma.

Lemma 6.1 (*[3, Corollary 8.4.4]*). *With reference to Assumption 5.1, there exist $\alpha_0, \alpha_1 \in \mathbb{C}$ such that each of θ_i, θ^*_i is $\alpha_0 + \alpha_1 q^{D-2i}$ for $0 \leq i \leq D$. Moreover $\alpha_1 \neq 0$.*

Definition 6.2 (*[11, Definition 9.2]*). With reference to Notation 2.1 and Assumption 5.1, we define $A, A^* \in \mathrm{Mat}_X(\mathbb{C})$ so that

$$A_1 = \alpha_0 I + \alpha_1 A,$$
$$A^*_1 = \alpha_0 I + \alpha_1 A^*,$$

where α_0, α_1 are from Lemma 6.1. Thus for $0 \leq i \leq D$ the space $E_i V$ (resp. $E^*_i V$) is an eigenspace of A (resp. A^*) with eigenvalue q^{D-2i}.

The following result is immediate from Lemma 6.1 and Definition 6.2.

Lemma 6.3. *With reference to Assumption 5.1 and Definition 6.2, the following* (i), (ii) *hold.*

(i) $A = \sum_{i=0}^{D} q^{D-2i} E_i$.
(ii) $A^* = \sum_{i=0}^{D} q^{D-2i} E_i^*$.

Definition 6.4 (*[11, Definition 10.4]*). With reference to Definition 3.13 and Assumption 5.1, we define B, B^*, K, K^*, Φ, Ψ to be the unique matrices in $\mathrm{Mat}_X(\mathbb{C})$ that satisfy the requirements of the following table for $0 \le i, j \le D$.

The matrix	is 0 on
$B - q^{i-j} I$	$\tilde{V}_{i,j}^{\downarrow\uparrow}$
$B^* - q^{j-i} I$	$\tilde{V}_{i,j}^{\uparrow\downarrow}$
$K - q^{i-j} I$	$\tilde{V}_{i,j}^{\downarrow\downarrow}$
$K^* - q^{i-j} I$	$\tilde{V}_{i,j}^{\uparrow\uparrow}$
$\Phi - q^{i+j-D} I$	$\tilde{V}_{i,j}^{\downarrow\downarrow}$
$\Psi - q^{i+j-D} I$	$\tilde{V}_{i,j}^{\downarrow\uparrow}$

Proposition 6.5 (*[11, Lemma 12.1]*). *With reference to Assumption 5.1, the following* (i), (ii) *hold.*

(i) *Each of the matrices from the list (15) is contained in T.*
(ii) *Each of Φ, Ψ is central in T.*

Proposition 6.6. *With reference to Assumption 5.1 and Definition 6.4, the following* (i)–(vi) *hold.*

(i) $B = \sum_{i=0}^{D} \sum_{j=0}^{D} q^{i-j} E_{i,j}^{\downarrow\uparrow}$.
(ii) $B^* = \sum_{i=0}^{D} \sum_{j=0}^{D} q^{j-i} E_{i,j}^{\uparrow\downarrow}$.
(iii) $K = \sum_{i=0}^{D} \sum_{j=0}^{D} q^{i-j} E_{i,j}^{\downarrow\downarrow}$.
(iv) $K^* = \sum_{i=0}^{D} \sum_{j=0}^{D} q^{i-j} E_{i,j}^{\uparrow\uparrow}$.
(v) $\Phi = \sum_{i=0}^{D} \sum_{j=0}^{D} q^{i+j-D} E_{i,j}^{\downarrow\downarrow}$.
(vi) $\Psi = \sum_{i=0}^{D} \sum_{j=0}^{D} q^{i+j-D} E_{i,j}^{\downarrow\uparrow}$.

Proof. (i) Let $\check{B} = \sum_{i=0}^{D} \sum_{j=0}^{D} q^{i-j} E_{i,j}^{\downarrow\uparrow}$. We show that $B = \check{B}$. Using Definition 4.1 we find that $\check{B} - q^{i-j} I$ is 0 on $\tilde{V}_{i,j}^{\downarrow\uparrow}$ for $0 \le i, j \le D$, so $B = \check{B}$ in view of Definition 6.4.
(ii)–(vi) Similar to the proof of (i). $\square$

Lemma 6.7. *With reference to Assumption 5.1 and Definition 6.4, the following* (i)–(iv) *hold.*

(i) $\Phi = \sum_{\eta=0}^{D} q^{\eta} \phi_{\eta}$.
(ii) $\Phi = \sum_{i=0}^{D} \sum_{j=0}^{D} q^{D-i-j} E_{i,j}^{\uparrow\uparrow}$.
(iii) $\Psi = \sum_{\zeta=-D}^{D} q^{\zeta} \psi_{\zeta}$.
(iv) $\Psi = \sum_{i=0}^{D} \sum_{j=0}^{D} q^{D-i-j} E_{i,j}^{\uparrow\downarrow}$.

Proof. (i) Evaluate Proposition 6.6(v) using Corollary 4.3(i).
(ii) Evaluate (i) above using Corollary 4.3(ii).
(iii) Evaluate Proposition 6.6(vi) using Corollary 4.4(i).
(iv) Evaluate (iii) above using Corollary 4.4(ii). $\square$

7. The transpose of A, A^*, B, B^*, K, K^*, Φ, Ψ

In this section we find the transpose of each matrix in the list (15). We start with A and A^*.

Theorem 7.1. *With reference to Assumption 5.1 and Definition 6.2, the following* (i), (ii) *hold.*

(i) *A is symmetric.*
(ii) *A^* is symmetric.*

Proof. Immediate from Definition 6.2 and since each of A_1, A_1^* is symmetric. □

Theorem 7.2. *With reference to Assumption 5.1 and Definition 6.4, the following* (i), (ii) *hold.*

(i) $B^t = B^*$.
(ii) $K^t = K^{*-1}$.

Proof. (i) Combining Theorem 4.7(iii) and Proposition 6.6(i), (ii) we have

$$\begin{aligned} B^t &= \sum_{i=0}^{D} \sum_{j=0}^{D} q^{i-j} E^{\uparrow\downarrow}_{D-i,D-j} \\ &= \sum_{i=0}^{D} \sum_{j=0}^{D} q^{(D-j)-(D-i)} E^{\uparrow\downarrow}_{D-i,D-j} \\ &= \sum_{i=0}^{D} \sum_{j=0}^{D} q^{j-i} E^{\uparrow\downarrow}_{i,j} \\ &= B^*. \end{aligned}$$

(ii) Using Lemma 4.2 and Proposition 6.6(iv) we find $K^{*-1} = \sum_{i=0}^{D} \sum_{j=0}^{D} q^{j-i} E^{\uparrow\uparrow}_{i,j}$. Combining this with Theorem 4.7(ii) and Proposition 6.6(iii) we obtain

$$\begin{aligned} K^t &= \sum_{i=0}^{D} \sum_{j=0}^{D} q^{i-j} E^{\uparrow\uparrow}_{D-i,D-j} \\ &= \sum_{i=0}^{D} \sum_{j=0}^{D} q^{(D-j)-(D-i)} E^{\uparrow\uparrow}_{D-i,D-j} \\ &= \sum_{i=0}^{D} \sum_{j=0}^{D} q^{j-i} E^{\uparrow\uparrow}_{i,j} \\ &= K^{*-1}. \quad \square \end{aligned}$$

Theorem 7.3. *With reference to Assumption 5.1 and Definition 6.4, the following* (i), (ii) *hold.*

(i) *Φ is symmetric.*
(ii) *Ψ is symmetric.*

Proof. (i) Immediate from Corollary 4.8(ii) and Lemma 6.7(i).
(ii) Immediate from Corollary 4.9(ii) and Lemma 6.7(iii). □

We finish this section with a comment.

Proposition 7.4. *With reference to Definition 4.1 and Assumption 5.1, the following* (i), (ii) *hold.*

(i) $\Phi^{-1} = \sum_{i=0}^{D} \sum_{j=0}^{D} q^{i+j-D} E_{i,j}^{\uparrow\uparrow}$.
(ii) $\Psi^{-1} = \sum_{i=0}^{D} \sum_{j=0}^{D} q^{i+j-D} E_{i,j}^{\uparrow\downarrow}$.

Proof. (i) Define $\Lambda = \sum_{i=0}^{D} \sum_{j=0}^{D} q^{i+j-D} E_{i,j}^{\uparrow\uparrow}$. Using Lemmas 4.2 and 6.7(ii) we routinely find $\Phi\Lambda = I$, so $\Phi^{-1} = \Lambda$.
(ii) Similar to the proof of (i). $\square$

8. The complex conjugate of A, A^*, B, B^*, K, K^*, Φ, Ψ

In this section we find the complex conjugate of each matrix in the list (15). We begin with a remark.

Remark 8.1. Recall from Assumption 5.1 that $q \in \mathbb{C}$ satisfies $b = q^2$. Define $q' = -q$ and note that $(q')^2 = b$. There are two cases to consider. For $b > 1$ we have $\overline{q} = q$ and $\overline{q'} = q'$. For $b < -1$ we have $q \in i\mathbb{R}$ so $\overline{q} = q'$. For each matrix S obtained using q let S' denote the corresponding matrix associated with q'.

Lemma 8.2. *With reference to Assumption 5.1 and Definition 6.2, the following* (i)–(iv) *hold.*

(i) *If D is even then $A = A'$.*
(ii) *If D is odd then $A = -A'$.*
(iii) *If D is even then $A^* = A^{*\prime}$.*
(iv) *If D is odd then $A^* = -A^{*\prime}$.*

Proof. (i), (ii) By Lemma 6.3(i) and Remark 8.1 we find

$$
\begin{aligned}
A' &= \sum_{i=0}^{D} (q')^{D-2i} E_i \\
&= \sum_{i=0}^{D} (-q)^{D-2i} E_i .
\end{aligned}
$$

Comparing this with Lemma 6.3(i) we get the result.
(iii), (iv) Similar to the proof of (i), (ii) above. $\square$

Theorem 8.3. *With reference to Assumption 5.1 and Definition 6.2, 6.4, the following* (i), (ii) *hold.*

(i) *Assume $b > 1$. Then each of A, A^*, B, B^*, K, K^*, Φ, Ψ is real.*
(ii) *Assume $b < -1$. Then*

$$\overline{A} = A', \qquad \overline{A^*} = A^{*\prime}, \tag{16}$$
$$\overline{B} = B', \qquad \overline{B^*} = B^{*\prime}, \tag{17}$$
$$\overline{K} = K', \qquad \overline{K^*} = K^{*\prime}, \tag{18}$$
$$\overline{\Phi} = \Phi', \qquad \overline{\Psi} = \Psi'. \tag{19}$$

Proof. (i) Each of A, A^* is real by Lemma 6.3, since q is real and each of E_i, E_i^* is real for $0 \leq i \leq D$. For the remaining matrices the results are immediate from Theorem 4.7(i), Proposition 6.6, and Remark 8.1.
(ii) We first obtain the equation on the left in (16). By Lemma 6.3(i) and Remark 8.1 we find

$$\begin{aligned} A' &= \sum_{i=0}^{D} (q')^{D-2i} E_i \\ &= \sum_{i=0}^{D} (\overline{q})^{D-2i} E_i \end{aligned}$$

and this equals $\overline{A}$ since E_i is real for $0 \leq i \leq D$. We have now obtained the equation on the left in (16). The equation on the right in (16) is similarly obtained. Next we obtain the equation on the left in (17). By Theorem 4.7(i), Proposition 6.6(i), and Remark 8.1 we have

$$\begin{aligned} B' &= \sum_{i=0}^{D} \sum_{j=0}^{D} (q')^{i-j} E_{i,j}^{\downarrow\uparrow} \\ &= \sum_{i=0}^{D} \sum_{j=0}^{D} (\overline{q})^{i-j} E_{i,j}^{\downarrow\uparrow} \\ &= \overline{B}. \end{aligned}$$

The remaining equations are similarly obtained. $\square$

9. The q-tetrahedron algebra $\boxtimes_q$

Referring to Assumption 5.1, Ito and Terwilliger displayed an action of the q-tetrahedron algebra $\boxtimes_q$ on the standard module V of Γ [11]. In this section we compute the transpose and complex conjugate for the action of each generator of $\boxtimes_q$ on V. We start with the definition of $\boxtimes_q$.

For a nonzero scalar $q \in \mathbb{C}$ such that $q^2 \neq 1$ we define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad n = 0, 1, 2, \ldots$$

We let $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ denote the cyclic group of order 4.

Definition 9.1 (*[10, Definition 10.1]*). Let $\boxtimes_q$ denote the unital associative $\mathbb{C}$-algebra that has generators

$$\{x_{ij} \mid i, j \in \mathbb{Z}_4,\ j - i = 1 \text{ or } j - i = 2\}$$

and the following relations:

(i) For $i, j \in \mathbb{Z}_4$ such that $j - i = 2$,

$$x_{ij} x_{ji} = 1.$$

(ii) For $h, i, j \in \mathbb{Z}_4$ such that the pair $(i - h, j - i)$ is one of $(1, 1)$, $(1, 2)$, $(2, 1)$,

$$\frac{q x_{hi} x_{ij} - q^{-1} x_{ij} x_{hi}}{q - q^{-1}} = 1.$$

(iii) For $h, i, j, k \in \mathbb{Z}_4$ such that $i - h = j - i = k - j = 1$,

$$x_{hi}^3 x_{jk} - [3]_q x_{hi}^2 x_{jk} x_{hi} + [3]_q x_{hi} x_{jk} x_{hi}^2 - x_{jk} x_{hi}^3 = 0.$$

We call $\boxtimes_q$ the *q-tetrahedron algebra* or "q-tet" for short.

Proposition 9.2 (*[11, Theorem 11.1]*). *With reference to Assumption 5.1, there exists a $\boxtimes_q$-module structure on V such that the generators x_{ij} act as follows:*

Generator	x_{01}	x_{12}	x_{23}	x_{30}	x_{02}	x_{20}	x_{13}	x_{31}
Action on V	$A\Phi\Psi^{-1}$	$B\Phi^{-1}$	$A^*\Phi\Psi$	$B^*\Phi^{-1}$	$K\Psi^{-1}$	ΨK^{-1}	$K^*\Psi$	$\Psi^{-1}K^{*-1}$

Theorem 9.3. *With reference to Assumption 5.1 and Proposition 9.2, the following hold on V.*

$$x_{01}^t = x_{01}, \qquad x_{12}^t = x_{30}, \qquad x_{23}^t = x_{23}, \qquad x_{30}^t = x_{12},$$
$$x_{02}^t = x_{31}, \qquad x_{20}^t = x_{13}, \qquad x_{13}^t = x_{20}, \qquad x_{31}^t = x_{02}.$$

Proof. Combine Proposition 6.5(ii), Theorems 7.1–7.3, and Proposition 9.2. □

Theorem 9.4. *With reference to Assumption 5.1 and Proposition 9.2, the following* (i), (ii) *hold on V.*

(i) *Assume $b > 1$. Then $\overline{x_{ij}} = x_{ij}$ for each generator x_{ij} of $\boxtimes_q$.*
(ii) *Assume $b < -1$. Then $\overline{x_{ij}} = x'_{ij}$ for each generator x_{ij} of $\boxtimes_q$.*

Proof. (i) Immediate from Theorem 8.3(i) and Proposition 9.2.
(ii) Immediate from Theorem 8.3(ii) and Proposition 9.2. □

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